TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 304, Number 1, November 1987

VISIBILITY AND RANK ONE IN HOMOGENEOUS SPACES OF $K \leqslant 0$

MARÍA J. DRUETTA

ABSTRACT. In this paper we study relationships between the visibility axiom and rank one in homogeneous spaces of nonpositive curvature. We obtain a complete classification (in terms of rank) of simply connected homogeneous spaces of nonpositive curvature and dimension $\leqslant 4$. We provide examples, in every dim $\geqslant 4$, of simply connected, irreducible homogeneous spaces ($K \leqslant 0$) which are neither visibility manifolds nor symmetric spaces.

Introduction. This paper was motived by the following question: What relations are there between the visibility axiom and rank one in homogeneous spaces of nonpositive curvature? We give a complete answer to this problem in dimension $\leqslant 4$, which led us to obtain a complete classification (in terms of rank) of simply connected homogeneous spaces of nonpositive curvature and dimension $\leqslant 4$. We were also interested in knowing if a simply connected and irreducible homogeneous space of nonpositve curvature was necessarily a visibility manifold or a symmetric space of noncompact type. We provide examples showing that the above is not true in general.

Since a riemannian homogeneous manifold H of sectional curvature $K \leqslant 0$ admits a simply transitive and solvable Lie group of isometries, it can be represented as a solvable Lie group G with a left invariant metric of nonpositive curvature, and the relations and examples mentioned above are investigated in this context. We recall that the Lie groups G endowed with left invariant metrics of nonpositive curvature satisfying visibility were characterized in [**4**, Theorem 2.3 and Corollary 2.3], where it is proved that visibility is equivalent to the existence on G of a left invariant metric of negative curvature.

The organization of the paper is as follows. In §1 we describe the space of parallel Jacobi fields along a one-parameter subgroup of G which is a geodesic in terms of algebraic conditions on the Lie algebra $\mathfrak{g}$ of G (Theorem 1.4).

In §2, relationships between visibility and rank one are obtained. If $[\mathfrak{g}, \mathfrak{g}]$ is abelian and has codimension one in $\mathfrak{g}$, visibility and rank one are equivalent. In dimension $\leqslant 4$, visibility implies rank one (Theorems 2.4 and 2.6). We improve a result obtained in [**4**, Theorem 2.3] related to the characterization of the Lie groups

Received by the editors November 8, 1985 and, in revised form, August 14, 1986.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 53C30; Secondary 53C25.
Key words and phrases. Homogeneous spaces, visibility manifolds, rank.

with left invariant metrics of nonpositive curvature satisfying visibility (Theorem 2.2).

§3 is devoted to examples which complement the results stated above (§2). We exhibit in every dim $\geqslant 4$ examples of rank one simply connected homogeneous spaces which are not visibility manifolds. Moreover these spaces are irreducible and not symmetric.

Finally, in §4 we classify the simply connected homogeneous spaces H of nonpositive curvature and $\dim H \leqslant 4$: either H has rank one or H is isometric to $\mathbf{R} \times T^2$, $\mathbf{R} \times T^3$, $\mathbf{R}^2 \times T^2$, $H^2 \times T^2$, where T^3 is a rank one homogeneous space of dimension three satisfying visibility and H^2, T^2 are two-dimensional spaces of constant negative curvature (Theorem 4.2). In particular if H is an irreducible homogeneous space of nonpositive curvature and rank $\geqslant 2$, it must then have dimension $\geqslant 5$ (Corollary 4.4).

Preliminaries. Let M be a complete, simply connected riemannian manifold of nonpositive curvature ($K \leqslant 0$). M satisfies the *visibility axiom* if given p in M and $\varepsilon > 0$ there exists a number $r = r(p, \varepsilon) > 0$ such that if σ: $[a, b] \to M$ is a geodesic segment satisfying $d(p, \sigma) \geqslant r$ then $\measuredangle_p(\sigma(a), \sigma(b)) \leqslant \varepsilon$, where $\measuredangle$ denotes the angle subtended at p by $\sigma(a)$ and $\sigma(b)$, and d is the intrinsic distance induced by the metric in M. Such an M is a *visibility manifold*.

If γ is a unit speed geodesic in M, J_γ^P will denote the space of all parallel Jacobi fields along γ and rank(γ) is defined to be the dimension of the vector space J_γ^P. Let $J_{\gamma^\perp}^P$ be the subspace of J_γ^P of all parallel Jacobi fields which are orthogonal on γ. It is clear that $J_\gamma^P = \mathbf{R}\gamma' \oplus J_{\gamma^\perp}^P$ and hence $\operatorname{rank}(\gamma) = 1 + \dim J_{\gamma^\perp}^P$. We recall that a parallel vector field J along a geodesic γ is a Jacobi field if and only if $K(J(t), \gamma'(t)) = 0$ for all t in $\mathbf{R}$; this is immediate since for each p in M the linear operator in T_pM, $X \to R(X, Y)Y$ is symmetric and negative semidefinite (R denotes the curvature tensor).

The minimum of rank(γ) over all geodesics γ of M is called the *rank of* M and denoted by rank(M). (This definition was introduced in [**2**].) Clearly $1 \leqslant \operatorname{rank}(M) \leqslant \dim M$ and M is flat if and only if $\operatorname{rank}(M) = \dim M$. It also follows that $\operatorname{rank}(M_1 \times M_2) = \operatorname{rank}(M_1) + \operatorname{rank}(M_2)$ and it is important that the definition of rank(M) above coincides with the usual one if M is a symmetric space. In fact, let G be a noncompact semisimple Lie group with center finite and T a maximal compact subgroup of G such that $M = G/T$. Let $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$ relative to $\mathfrak{t}$ and $\mathfrak{a}$ an abelian maximal subspace of $\mathfrak{p}$; $\mathfrak{p}$ may be identified with the tangent space to M at $p = eT$ and for all X, Y orthonormal in $\mathfrak{p}$, $K(X, Y) = \langle [[X, Y], Y], X \rangle$ where $\langle\ ,\ \rangle$ is a metric in $\mathfrak{p}$ that in the irreducible case is up to a positive constant the Killing form on $\mathfrak{g}$ restricted to $\mathfrak{p}$. Then if $\gamma(t) = (\exp tX)\, T$ is the geodesic in M with $\gamma'(0) = X$ ($X \in \mathfrak{p}$) J_γ^P coincides with $\mathscr{C}_\mathfrak{p}(X)$, the centralizer of X in $\mathfrak{p}$; hence rank(M) will be the minimum of $\dim \mathscr{C}_\mathfrak{p}(X)$ with X in $\mathfrak{p}$. Since $\dim \mathscr{C}_\mathfrak{p}(X) \geqslant \dim \mathfrak{a}$ for all X in $\mathfrak{p}$, by choosing Z in $\mathfrak{a}$ such that $\mathscr{C}_\mathfrak{p}(Z) = \mathfrak{a}$ (for instance a regular element in $\mathfrak{a}$) it follows that rank(M) defined as above coincides with $\dim \mathfrak{a}$, which is the usual one in a symmetric space (see [7]).

H will denote a simply connected homogeneous riemannian manifold of nonpositive curvature ($K \leqslant 0$). Since H is homogeneous rank(H) is the minimum of rank(γ) over all unit speed geodesic γ of H such that $\gamma(0) = p$ *for some* p *in* H. If H has sectional curvature $K < 0$ then there exists a constant C such that $K \leqslant C < 0$, so H satisfies the visibility (uniform) axiom (see **[5]**) and it follows also that H has rank one.

Since H is homogeneous, it admits a simply transitive solvable group of isometries, hence H is isometric to a solvable Lie group with a left invariant metric of nonpositive curvature.

Given a Lie group G with Lie algebra $\mathfrak{g}$ and left invariant metric $\langle\ ,\ \rangle$ (that is for all g in G the left translations Lg: $G \to G$, $x \to gx$ are isometries) we recall that if $X, Y, Z \in \mathfrak{g}$ then the riemannian connection ∇ is given by

$$2\langle \nabla_X Y, Z\rangle = \langle [X, Y], Z\rangle - \langle [Y, Z], X\rangle + \langle [Z, X], Y\rangle,$$

∇_X: $\mathfrak{g} \to \mathfrak{g}$ is skew-symmetric and satisfies $\nabla_X Y - \nabla_Y X = [X, Y]$ (torsion free).

Set $U(X, Y) = (\mathrm{ad}_X)^* Y + (\mathrm{ad}_Y)^* X$ where $(\mathrm{ad}_X)^*$ denotes the adjoint transformation of ad_X, then $2\nabla_X Y = \mathrm{ad}_X Y - U(X, Y)$ and the sectional curvature K reduces to

$$\begin{aligned} |X \wedge Y|^2 K(X, Y) &= \langle R(X, Y)Y, X\rangle \\ &= \tfrac{1}{4}|U(X, Y)|^2 - \tfrac{1}{4}\langle U(X, X), U(Y, Y)\rangle \\ &\quad -\tfrac{3}{4}|[X, Y]|^2 - \tfrac{1}{2}\langle [[X, Y], Y], X\rangle \\ &\quad -\tfrac{1}{2}\langle [[Y, X], X], Y\rangle \quad \text{where } R(X, Y) = [\nabla_X, \nabla_X] - \nabla_{[X, Y]}. \end{aligned}$$

A subalgebra $\mathfrak{a}$ of $\mathfrak{g}$ is said to be *totally geodesic* if $X, Y \in \mathfrak{a}$ implies $\nabla_X Y \in \mathfrak{a}$. The connected Lie subgroup A with Lie algebra $\mathfrak{a}$ is a totally geodesic submanifold of G. Observe that $A = \exp(\mathfrak{a})$ where exp: $\mathfrak{g} \to G$ is the exponential mapping of G. If $\mathfrak{a}$ is an abelian and totally geodesic subalgebra of $\mathfrak{g}$ then $\nabla_X Y = 0$ for all $X, Y \in \mathfrak{a}$. Theorem 5.2 of **[1]** shows that if $\mathfrak{g}$ is solvable then $\mathfrak{a}$, the orthogonal complement of $[\mathfrak{g}, \mathfrak{g}]$ in $\mathfrak{g}$, is an abelian subalgebra and hence totally geodesic since $[\mathfrak{g}, \mathfrak{g}]$ is an ideal of $\mathfrak{g}$.

For details on the subject the reader is referred to **[1, 2, 5, 6]**.

1. Parallel Jacobi vector fields. Let G be a Lie group of dimension n and $\mathfrak{g}$ be its Lie algebra. Let $\langle\ ,\ \rangle$ be a left invariant metric on G and $\{X_1, \ldots, X_n\}$ be an orthonormal basis for $\mathfrak{g}$.

If γ is a unit speed geodesic in G and J is a Jacobi vector field along γ, then J is a solution of the Jacobi equation

$$\nabla_{\gamma'}(\nabla_{\gamma'} J) - R(\gamma', J)\gamma' = 0.$$

If f_i are the components of J with respect to the basis $\{X_i\}_{i=1}^n$, then $J(t) = (dL_{\gamma(t)})_e(X(t))$ where $X(t) = \sum_{i=1}^n f_i(t) X_i$.

We recall that not every geodesic starting at the identity in G can be written $\gamma(t) = \exp tZ$ for some $Z \in \mathfrak{g}$. However such one-parameter curves are geodesics if $Z \in [\mathfrak{g}, \mathfrak{g}]^{\perp}$, the orthogonal complement of $[\mathfrak{g}, \mathfrak{g}]$ in $\mathfrak{g}$.

For $\gamma(t) = \exp tZ$ the Jacobi equation becomes

$$\sum_{i=1}^{n} \{ f_i''(t) X_i + 2f_i'(t)\nabla_Z X_i + f_i(t)[\nabla_Z(\nabla_Z X_i) - R(Z, X_i)Z] \} = 0.$$

Furthermore from the curvature formula and the facts $\nabla_Z Z = 0$, ∇ torsion free, we obtain

$$\begin{aligned}\nabla_Z(\nabla_Z X_i) - R(Z, X_i)Z &= \nabla_Z(\nabla_Z X_i) - \nabla_Z(\nabla_{X_i} Z) + \nabla_{[Z,X_i]}Z \\ &= \nabla_Z[Z, X_i] + \nabla_{[Z,X_i]}Z = 2\nabla_{[Z,X_i]}Z + [Z,[Z, X_i]] = -U([Z, X_i], Z).\end{aligned}$$

Therefore the Jacobi equation along γ simplifies to

$$(1) \qquad \sum_{i=1}^{n} \{ f_i''(t) X_i + 2f_i'(t)\nabla_Z X_i - f_i(t)U(Z,[Z, X_i]) \} = 0 \quad \text{for all } t.$$

If moreover J is a parallel field, that is $\nabla_{\gamma'} J = 0$, we get

$$(2) \qquad \sum_{i=1}^{n} \{ f_i'(t) X_i + f_i(t)\nabla_Z X_i \} = 0 \quad \text{for all } t.$$

Using the linearity of ∇_Z and $U(Z, [Z, \ ])$ the above equations in $\mathfrak{g}$ are equivalent to

$$(1') \qquad X''(t) + 2\nabla_Z X'(t) - U(Z, [Z, X(t)]) = 0,$$

$$(2') \qquad X'(t) + \nabla_Z X(t) = 0.$$

Differentiating (2′) and substituting it in (1′) it follows that J is a parallel Jacobi field if and only if

$$(3) \qquad X'(t) + \nabla_Z X(t) = 0, \quad \nabla_Z X'(t) - U(Z, [Z, X(t)]) = 0.$$

Let A and B be the $n \times n$ matrices for ad_Z and $U_Z = U(Z, \ )$, respectively, with respect to a given basis for $\mathfrak{g}$ and $S = \frac{1}{2}(A - B)$ the matrix for ∇_Z. If $f = (f_1, \ldots, f_n)$ are the coordinates of X then (3) is equivalent to

$$(3') \qquad f' + Sf = 0, \quad Sf' - BAf = 0.$$

The following lemma gives us a useful formula for the sectional curvature.

Lemma 1.1. *If X and Z are orthonormal vectors in $\mathfrak{g}$ and $\gamma(t) = \exp tZ$ is a geodesic, then*

$$K(X, Z) = -\langle (\nabla_Z^2 + U_Z \circ \operatorname{ad}_Z) X, X \rangle.$$

Proof. From the sectional curvature formula

$$\begin{aligned}K(X, Z) &= \tfrac{1}{4}|U_Z X|^2 - \tfrac{3}{4}|\operatorname{ad}_Z X|^2 - \tfrac{1}{2}\langle \operatorname{ad}_Z^2 X, X\rangle + \tfrac{1}{2}\langle \operatorname{ad}_Z X, \operatorname{ad}_X^* Z\rangle \\ &= \tfrac{1}{4}|U_Z X|^2 - \tfrac{3}{4}|\operatorname{ad}_Z X|^2 - \tfrac{1}{2}\langle \operatorname{ad}_Z^2 X, X\rangle \\ &\quad + \tfrac{1}{2}\langle \operatorname{ad}_Z X, U_Z X\rangle - \tfrac{1}{2}\langle \operatorname{ad}_Z X, \operatorname{ad}_Z^* X\rangle \\ &= \tfrac{1}{4}|U_Z X|^2 - \tfrac{3}{4}|\operatorname{ad}_Z X|^2 - \langle \operatorname{ad}_Z^2 X, X\rangle + \tfrac{1}{2}\langle \operatorname{ad}_Z X, U_Z X\rangle \\ &= \tfrac{1}{4}|U_Z X - \operatorname{ad}_Z X|^2 - |\operatorname{ad}_Z X|^2 - \langle \operatorname{ad}_Z^2 X, X\rangle + \langle \operatorname{ad}_Z X, U_Z X\rangle \\ &= |\nabla_Z X|^2 - |\operatorname{ad}_Z X|^2 - \langle \operatorname{ad}_Z^2 X, X\rangle + \langle (U_Z^* \circ \operatorname{ad}_Z) X, X\rangle \\ &= -\langle (\nabla_Z^2 + \operatorname{ad}_Z^* \circ \operatorname{ad}_Z + \operatorname{ad}_Z^2 - U_Z^* \circ \operatorname{ad}_Z) X, X\rangle.\end{aligned}$$

Since $2\nabla_Z = \mathrm{ad}_Z - U_Z$ then

$$\nabla_Z^2 + (\mathrm{ad}_Z^* - U_Z^*)\circ \mathrm{ad}_Z + \mathrm{ad}_Z^2 = \nabla_Z^2 + 2\nabla_Z^* \circ \mathrm{ad}_Z + \mathrm{ad}_Z^2$$
$$= \nabla_Z^2 - 2\nabla_Z \circ \mathrm{ad}_Z + \mathrm{ad}_Z^2 = \nabla_Z^2 + U_Z \circ \mathrm{ad}_Z$$

and the lemma follows.

REMARK 1.2. If the metric has $K \leqslant 0$ and X, Z are linearly independent in $\mathfrak{g}$ such that $[X, Z] = 0$ it follows from the previous lemma that $|\nabla_Z X|^2 = 0$. Hence by using (3) we get that $J(t) = (dL_{\gamma(t)})_e \cdot X$ is a parallel Jacobi field along $\gamma(t) = \exp tZ$ with $J(0) = X$.

In the sequel G will denote a solvable Lie group of dimension n with a left invariant metric of sectional curvature $K \leqslant 0$. We give next a description of the parallel Jacobi fields along the geodesic $\gamma(t) = \exp tZ$ in terms of the Lie algebra $\mathfrak{g}$ of G.

LEMMA 1.3. *Let $J(t) = (dL_{\gamma(t)})_e \cdot X(t)$, $X(t) \in \mathfrak{g}$. $J(t)$ is a parallel Jacobi field along γ if and only if $X(t)$ satisfies*

$$X'(t) + \nabla_Z X(t) = 0, \quad (\nabla_Z^2 + U_Z \circ \mathrm{ad}_Z) X(t) = 0, \tag{5}$$

or, equivalently, in the matricial form f satisfies

$$f'(t) + Sf(t) = 0, \quad (S^2 + BA) f(t) = 0. \tag{5'}$$

PROOF. Assume J is a parallel Jacobi field. Then X satisfies (3); hence by applying ∇_Z to the first equation we get $\nabla_Z X' + \nabla_Z^2 X = 0$ and $-\nabla_Z^2 X - U_Z \circ \mathrm{ad}_Z X = 0$. To prove the converse from (5) and Lemma 1.1, J is parallel and $K(J(t), \gamma'(t)) = 0$ for all t; consequently J is a parallel Jacobi field along γ.

In the sequel, we will not distinguish between the matrices A, B, S and the operators they represent. We observe that the solution of the equation $X' + SX = 0$ such that $X(0) = X$ is given by $X(t) = (\exp -tS)X$ where exp denotes the usual exponential map (or matrix), $\exp T = \sum_{n=0}^{\infty} T^n/n!$.

THEOREM 1.4. *Let $J(t) = (dL_{\gamma(t)})_e \cdot X(t)$ where $\gamma(t) = \exp tZ$.*

(i) *If J is a parallel Jacobi field along γ then $X(t) = (\exp -tS)X$ and X satisfies $(S^2 + BA)S^i X = 0$ for all integers $i \geqslant 0$.*

(ii) *Conversely, if $X \in \mathfrak{g}$ satisfies $(S^2 + BA)S^i X = 0$ for any integer $i \geqslant 0$, then $X(t) = (\exp -tS)X$ gives the parallel Jacobi field J along γ with $J(0) = X$.*

PROOF. Assume that J is a parallel Jacobi field. Then by Lemma 1.3 $X(t) = (\exp -tS)X$ with $X \in \mathfrak{g}$, and $(S^2 + BA)(\exp -tS)X = 0$ for all $t \in \mathbf{R}$; taking derivatives of order i at $t = 0$ it follows that $(S^2 + BA)S^i X = 0$ for any integer $i \geqslant 0$.

To prove (ii) we observe first that from $(S^2 + BA)X = 0$ we obtain

$$(S^2 + BA)(\exp -tS)X = (\exp -tS)S^2 X + BA(\exp -tS)X = [BA, \exp -tS]X,$$

and $[BA, S^i]X = (S^2 + BA)S^i X$ for any integer $i \geqslant 0$. Then

$$(S^2 + BA)(\exp -tS)X = [BA, \exp -tS]X$$
$$= \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} t^i [BA, S^i] X = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} t^i (S^2 + BA) S^i X$$

and consequently $(S^2 + BA)S^iX = 0$ for all $i \geqslant 0$ implies that

$$(S^2 + BA)(\exp -tS)X = 0 \quad \text{for all } t \in \mathbf{R}.$$

Now (ii) follows by applying Lemma 1.3.

If J_Z^P denotes the space of parallel Jacobi fields along $\exp tZ$, we have

COROLLARY 1.5. *J_Z^P is isomorphic to the S-invariant subspace of $\mathfrak{g}$ defined by $\mathfrak{p} = \{X \in \mathfrak{g}\colon (S^2 + BA)S^iX = 0,\ i = 0, \dots, n\}$. In particular $1 \leqslant \dim J_Z^P \leqslant n -$ rank$(S^2 + BA)$.*

PROOF. Since any parallel Jacobi field J is completely determined by $J(0)$, it follows from Theorem 1.4 that J_Z^P is isomorphic to the subspace of $\mathfrak{g}$ defined by $(S^2 + BA)S^iX = 0$ for any integer $i \geqslant 0$. We only need to show that this subspace coincides with $\mathfrak{p}$. In fact, since the subset of $\mathfrak{g}$, $\{S^iX\colon i = 0, \dots, n\}$ is linearly dependent, there exists a nontrivial linear combination $\sum_{i=0}^n a_iS^iX = 0$. If $r = \max\{i\colon a_i \neq 0\}$ then $S^rX = \sum_{i=0}^{r-1} b_iS^iX$, $r \leqslant n$, and an easy computation shows that $(S + BA)S^iX = 0$, $i = 0, \dots, r$, implies $(S^2 + BA)S^iX = 0$ for any integer $i \geqslant 0$. Since $\mathfrak{p}$ is clearly invariant under S, the corollary follows.

REMARK 1.6. When $Z \in \mathfrak{a}$, the orthogonal complement of $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ in $\mathfrak{g}$, $\exp tZ$ is geodesic, and since $\mathfrak{a}$ is abelian it follows from Remark 1.2 that

$$J_Z^P \cong \mathfrak{a} + \{X \in \mathfrak{g}'\colon (S^2 + BA)S^iX = 0,\ i = 0, \dots, \dim \mathfrak{g}'\}$$

and hence

$$\dim J_Z^P = \dim \mathfrak{a} + \dim\{X \in \mathfrak{g}'\colon (S^2 + BA)S^iX = 0,\ i = 0, \dots, \dim \mathfrak{g}'\}$$

Note that in this case, $B = (\mathrm{ad}_Z)^*$, S is the skew-symmetric part of ad_Z, and $S^2 + BA$ is a symmetric and positive semidefinite operator on $\mathfrak{g}'$, such that $-\langle (S^2 + BA)X, X\rangle = K(X, Z)$.

2. Visibility and rank one. In this section we assume G is a solvable and simply connected Lie group with a left invariant metric $\langle\ ,\ \rangle$ of sectional curvature $K \leqslant 0$ and Lie algebra $\mathfrak{g} = \mathfrak{g}' \oplus \mathbf{R}e_1$, e_1 a unit vector in $\mathfrak{g}'^{\perp}$, the orthogonal complement of $\mathfrak{g}'$ in $\mathfrak{g}$. $\mathfrak{z}$ will denote the center of $\mathfrak{g}'$ ($\mathfrak{z} \neq 0$ since $\mathfrak{g}'$ is nilpotent) and $D = \frac{1}{2}(\mathrm{ad}_{e_1} + ad_{e_1}^*)$ the symmetric part of $\mathrm{ad}_{e_1}|_{\mathfrak{g}'}$ with respect to the metric $\langle\ ,\ \rangle$.

PROPOSITION 2.1. *D is positive semidefinite; that is $\langle DX, X\rangle \geqslant 0$ (or $\leqslant 0$) for any X in $\mathfrak{g}'$*

PROOF. We observe first that for Z, X in $\mathfrak{g}'\langle U(Z, X), e_1\rangle = -2\langle DZ, X\rangle$, then

$$(*)\quad |X \wedge Z|^2 K(X, Z) = \tfrac{1}{4}\left|U(Z, X)|_{\mathfrak{g}'}\right|^2 + \langle DZ, X\rangle^2 - \langle DZ, Z\rangle\langle DX, X\rangle.$$

Now we will show that if $\langle DZ, Z\rangle = 0$ for all $Z \in \mathfrak{z}$ then $\mathfrak{g}'$ is abelian and hence $D|_{\mathfrak{g}'} = 0$. In fact, under this assumption, by (*) $(\mathrm{ad}_X)^*Z|_{\mathfrak{g}'} = 0$ for Z in $\mathfrak{z}$ and X in $\mathfrak{g}'$ since $K \leqslant 0$. Hence $\langle Z, [X, Y]\rangle = 0$ for all $Z \in \mathfrak{z}$, X, Y in $\mathfrak{g}'$ and consequently $[\mathfrak{g}', \mathfrak{g}']$ is orthogonal to $\mathfrak{z}$ in $\mathfrak{g}'$; if $[\mathfrak{g}', \mathfrak{g}'] \neq 0$ we get a contradiction from the fact that $\mathfrak{z} \cap [\mathfrak{g}', \mathfrak{g}'] \neq 0$.

If there exists Z in z such that $\langle DZ, Z\rangle > 0$ (or < 0) then $\langle DZ, Z\rangle\langle DX, X\rangle \geqslant \frac{1}{4}|\mathrm{ad}_X^* Z|^2 + \langle DZ, X\rangle^2 \geqslant 0$ (see (*)) implies $\langle DX, X\rangle \geqslant 0$ (or $\leqslant 0$) for any X in $\mathfrak{g}'$, finishing the proof.

In [**4**, Theorem 2.3] we showed that the visibility axiom is equivalent to D being positive definite on z, but actually we can say more.

THEOREM 2.2. *If G satisfies visibility then D is definite on $\mathfrak{g}'$.*

PROOF. We assume $\mathfrak{g} = \mathfrak{g}' + \mathbf{R}e_1$ with e_1 a unit vector in $\mathfrak{g}'^{\perp}$ such that D is positive semidefinite.

For $\varepsilon > 0$ let $(\mathfrak{g}_\varepsilon, \langle\ ,\ \rangle)$ be the Lie algebra with an inner product which is the same as the one in $\mathfrak{g}$ and Lie bracket $[\ ,\ ]_\varepsilon$ defined by

$$[X, Y]_\varepsilon = \varepsilon[X, Y], \quad [e_1, X]_\varepsilon = [e_1, X] \quad \text{for all } X, Y \text{ in } \mathfrak{g}'.$$

We observe that $\mathfrak{g}'_\varepsilon = \mathfrak{g}'$; D and S, the symmetric and skew-symmetric part of $\mathrm{ad}_{e_1}|_{\mathfrak{g}'}$, respectively, coincide with D_ε and S_ε, the symmetric and skew-symmetric part of $\mathrm{ad}_{e_1}|_{\mathfrak{g}'_\varepsilon}$ respectively.

If K_ε denotes the sectional curvature in $(\mathfrak{g}_\varepsilon, \langle\ ,\ \rangle)$ a straightforward computation shows

$$K_\varepsilon(X, Y) = \varepsilon^2 K(X, Y) + (1 - \varepsilon^2)\left[\langle DX, Y\rangle^2 - \langle DX, X\rangle\langle DY, Y\rangle\right],$$
$$K_\varepsilon(X, e_1) = -\langle (D^2 + [D, S])X, X\rangle \quad \text{for any } X, Y \text{ in } \mathfrak{g}'.$$

Let $\{e_1, e_2, \ldots, e_n\}$ be an orthonormal basis for $\mathfrak{g}$. The structure constants $\alpha_{ijk}(\varepsilon)$ defined by $[e_i, e_j]_\varepsilon = \sum_{k=1}^n \alpha_{ijk}(\varepsilon)e_k$ tend to well-defined limits as $\varepsilon \to 0$. Thus we obtain a limit algebra $\mathfrak{g}_0$ whose $\mathfrak{g}'_0$ is abelian and with prescribed metric $\langle\ ,\ \rangle$. It follows by continuity that K_0, the sectional curvature associated to $\mathfrak{g}_0$, is nonpositive. In fact, if $X, Y \in \mathfrak{g}'$,

$$K_0(X, Y) = \lim_{\varepsilon \to 0} K_\varepsilon(X, Y) = \langle DX, Y\rangle^2 - \langle DX, X\rangle\langle DY, Y\rangle \leqslant 0$$

since D is positive semidefinite on $\mathfrak{g}'$, and

$$K_0(X, e_1) = \lim_{\varepsilon \to 0} K_\varepsilon(X, e_1) = K(X, e_1) \leqslant 0.$$

Take an arbitrary two-dimensional subspace $\Pi = \{\alpha X + \beta e_1, Y\}$ of $\mathfrak{g}$, where X and Y are orthonormal vectors of $\mathfrak{g}'$ and $\alpha^2 + \beta^2 = 1$. One computes

$$K_0(\alpha X + \beta e_1, Y) = \alpha^2 K_0(X, Y) + \beta^2 K_0(e_1, Y) - 2\alpha\beta\langle R_0(X, Y)e_1, Y\rangle.$$

Since $\mathfrak{g}'_0$ is abelian, $\langle R_0(X, Y)e_1, Y\rangle = 0$ and consequently $K_0 \leqslant 0$.

By hypothesis G satisfies visibility. Then [**4**, Theorem 2.3] ensures that $\mathrm{ad}_{e_1}|_{\mathfrak{g}'}$ has all eigenvalues with nonzero real parts. From $\mathrm{ad}_{e_1^0}|_{\mathfrak{g}'_0} = \mathrm{ad}_{e_1}|_{\mathfrak{g}'}$ $(\mathfrak{g}'_0 = \mathfrak{g}')$ it follows also by [**4**, Theorem 2.3] that $(\mathfrak{g}_0, \langle\ ,\ \rangle)$ satisfies visibility and hence (again [**4**, Theorem 2.3]) D_0 is definite on $\mathfrak{g}'_0 = \mathfrak{g}'$ since $\mathfrak{g}'_0$ is abelian; this implies D is definite on $\mathfrak{g}'$ $(D_0 = \lim_{\varepsilon \to 0} D_\varepsilon = D)$.

COROLLARY 2.3. *Let G be a solvable Lie group with a left invariant metric of $K \leqslant 0$. G satisfies visibility if and only if $\mathfrak{g}'^{\perp} = \mathbf{R}e_1$, e_1 a unit vector and D, the symmetric part of* $\mathrm{ad}_{e_1}|_{\mathfrak{g}'}$ *is definite on $\mathfrak{g}'$. (Compare with* [**4**, *Theorem* 2.3].)

The following theorems gives us a relation between visibility axiom and rank one.

THEOREM 2.4. (i) *If G has rank one and $\mathfrak{g}'$ has codimension one in $\mathfrak{g}$ then G satisfies visibility.* (ii) *If $\mathfrak{g}'$ is abelian and G satisfies visibility then G has rank one.*

PROOF. (i) Assume the visibility axiom fails in G. Since $[G,G]$, the connected Lie subgroup of G, with Lie algebra $\mathfrak{g}'$ has codimension one in G it follows from [**4**, Theorem 3.1] that G is isometric to a Riemannian product $\mathbf{R}^k \times T$ with $k > 0$ and T satisfying visibility; hence G has rank at least two.

(ii) If G satisfies visibility then $\mathfrak{g} = \mathfrak{g}' + \mathbf{R}e_1$ where e_1 is a unit vector in $\mathfrak{g}'^{\perp}$ such that D is positive definite (Corollary 2.3). We observe that there exists a unit vector X in $\mathfrak{g}'$ such that $K(X, e_1) < 0$. In fact, if that fails, the Ricci curvature in the e_1 direction is zero; hence ad_{e_1} is skew-symmetric and $D = 0$ (see [**8**, Lemma 2.3]) contradicting visibility. We next show that $K(X, Z) < 0$ for all $Z \in \mathfrak{g}$ independent of X. In fact, for all Y orthonormal with X in $\mathfrak{g}'$, $K(X, Y) = \langle DX, Y\rangle^2 - \langle DX, X\rangle\langle DY, Y\rangle < 0$ (D is positive definite). Then, since $\mathfrak{g}'$ is abelian $(1 + \beta^2)K(X, Y + \beta e_1) = K(X, Y) + \beta^2 K(X, e_1) < 0$.

Therefore, there is no parallel Jacobi field J orthogonal on the geodesic γ in G such that $\gamma(0) = e$, $\gamma'(0) = X$ (in this case $K(X, J(0)) = 0$); consequently $\dim J_\gamma^P = 1$ and G has rank one.

COROLLARY 2.5. *If $\mathfrak{g}'$ is abelian and it has codimension one in $\mathfrak{g}$ then visibility, rank one and without de Rham flat factor are equivalent in G.*

PROOF. That visibility and rank one are equivalent is immediate from Theorem 2.4. If G has rank one it is clear that G is without flat factor in the de Rham decomposition, and if rank one (or visibility) fails in G then by [**4**, Theorem 3.1] G admits de Rham flat factor.

THEOREM 2.6. *Let G be a solvable Lie group with a left invariant metric of sectional curvature $K \leqslant 0$. If* $\dim G \leqslant 4$ *and G satisfies visibility then* $\mathrm{rank}(G) = 1$.

PROOF. We may assume that $\dim G = 4$ and $\mathfrak{g}'$ is nonabelian, since otherwise $\mathfrak{g}'$ is abelian and the result follows from Theorem 2.4.

Let $\mathfrak{g} = \mathfrak{g}' + \mathbf{R}e_1$ with $\mathfrak{g}'$ nonabelian and e_1 a unit vector in $\mathfrak{g}'^{\perp}$ such that D is positive definite. Hence $\dim \mathfrak{g}' = 3$ and $z = \mathbf{R}e_2$ with e_2 a unit vector in $\mathfrak{g}'$. Since $\det(\mathrm{ad}_{e_1}|_{\mathfrak{g}'}) \neq 0$ (otherwise visibility fails), $\mathrm{ad}_{e_1}(z^{\perp})$ is a two-dimensional subspace of $\mathfrak{g}'$ such that $\mathrm{ad}_{e_1}(z^{\perp}) \cap z \neq 0$. Then there exists an orthonormal basis of $\mathfrak{g}'$, $\{e_2, e_3, e_4\}$, such that

$$[e_1, e_2] = ae_2, \quad [e_1, e_3] = \alpha e_3 + \beta e_4, \quad [e_1, e_4] = be_2 + \gamma e_3 + \delta e_4.$$

$(*)$ with $a > 0$, $\alpha > 0$, $\delta > 0$. We will show next that $\dim J_{e_1}^P = 1$ and consequently G has rank one. We remark first that in general, under the visibility assumption there is no $X \neq 0$ in $\mathfrak{g}'$ such that $(S^2 + BA)X = 0$ and $SX = 0$. In fact, these conditions together imply $|AX|^2 = \langle BAX, X\rangle = 0$ and $DX = [e_1, X] = 0$ which contradicts visibility. As a consequence, since $\langle X, SX\rangle = 0$ it follows, from Remark 1.6, that $J_{e_1^{\perp}}^P$ being S-invariant cannot be one dimensional. ($J_{e_1^{\perp}}^P$ denotes the space of all parallel Jacobi fields which are orthogonal on $\exp te_1$.)

Let A, B and S be the matrices for $\mathrm{ad}_{e_1}|_{\mathfrak{g}'}$, $\mathrm{ad}^*_{e_1}|_{\mathfrak{g}'}$ and the skew-symmetric part of $\mathrm{ad}_{e_1}|_{\mathfrak{g}'}$, respectively with respect to the basis $\{e_2, e_3, e_4\}$. Then

$$A = \begin{vmatrix} a & 0 & b \\ 0 & \alpha & \gamma \\ 0 & \beta & \delta \end{vmatrix}, \quad B = \begin{vmatrix} a & 0 & 0 \\ 0 & \alpha & \beta \\ b & \gamma & \delta \end{vmatrix}, \quad S = \begin{vmatrix} 0 & 0 & \frac{b}{2} \\ 0 & 0 & \frac{\gamma - \beta}{2} \\ -\frac{b}{2} & \frac{\beta - \gamma}{2} & 0 \end{vmatrix}$$

and

$$S^2 + BA = \begin{vmatrix} -\frac{b^2}{4} + a^2 & \frac{b}{4}(\beta - \gamma) & ab \\ \frac{b}{4}(\beta - \gamma) & \alpha^2 + \beta^2 - \frac{(\beta - \gamma)^2}{4} & \alpha\gamma + \beta\delta \\ ab & \alpha\gamma + \beta\delta & \frac{3}{4}b^2 + \gamma^2 + \delta^2 - \frac{(\beta - \gamma)^2}{4} \end{vmatrix}.$$

Observe that since the first row of $S^2 + BA$ is nonzero, $\mathrm{rank}(S^2 + BA) \geqslant 1$.

If $\mathrm{rank}(S^2 + BA) = 1$, it is obvious that $b \neq 0$ (compare columns 1 and 2 of $S^2 + BA$). It follows that:

(i) The column space of $S^2 + BA$ is spanned by

$$X = \left(-b^2/4 + a^2, b(\beta - \gamma)/4, ab\right).$$

(ii) $\ker(S^2 + BA)$ has dimension two and since $S^2 + BA$ is symmetric the spaces $\mathbf{R}X$ and $\ker(S^2 + BA)$ are orthogonal complements.

(iii) S has rank four and $\ker S$ is spanned by $Y = (\beta - \gamma, b, 0)$.

Since $J^P_{e_1^\perp}$ is isomorphic to a subspace of $\ker(S^2 + BA)$ and cannot have dimension one, then $J^P_{e_1^\perp} = 0$ or $J^P_{e_1^\perp} = \ker(S^2 + BA)$. In the last case, $\ker(S^2 + BA)$ is S-invariant ($J^P_{e_1^\perp}$ is S-invariant) and consequently SX is a multiple of X. Hence, $SX = 0$ (S is skew-symmetric) and X is a multiple of Y, which is impossible since $ab \neq 0$. Therefore $J^P_{e_1^\perp} = 0$ and $\mathrm{rank}(G) = 1$.

If $\mathrm{rank}(S^2 + BA) = 2$ or 3 then $\ker(S^2 + BA)$ has dimension zero or one and from the remark above $J^P_{e_1^\perp}$ has dimension zero. Hence, $J^P_{e_1}$ has dimension one and $\mathrm{rank}(G) = 1$.

COROLLARY 2.7. *Let $H = G/T$ be a homogeneous space with a G-invariant metric of nonpositive curvature. If* $\dim H \leqslant 4$ *and H admits a G-invariant metric with negative curvature then H has rank one.*

PROOF. It is immediate from [**4**, Theorem 2.1] after recalling that H is isometric to a solvable Lie group with a left invariant metric of nonpositive curvature.

REMARK 2.8. Example 3.3 in §3 provides an example of a Lie group G of dimension $\geqslant 4$ with a left invariant metric of nonpositive curvature such that G does not satisfy visibility and $\mathrm{rank}(G) = 1$.

3. Examples. In this section we provide some examples of Lie algebras with metrics of nonpositive curvature and their respective ranks. In particular in Example 3.3 we exhibit a family of rank one homogeneous manifolds H not satisfying visibility. Moreover H is irreducible and not symmetric.

Let $\mathfrak{g}$ be a Lie algebra with an inner product $\langle\ ,\ \rangle$ such that $\mathfrak{a}$, the orthogonal complement of $\mathfrak{g}'$ in $\mathfrak{g}$ with respect to $\langle\ ,\ \rangle$ is abelian. If D_H, S_H denote the symmetric and skew-symmetric part of $\mathrm{ad}_H|_{\mathfrak{g}'}$ with H in $\mathfrak{a}$, it is an easy computation, using the connection formula, to show that $\nabla_H X = S_H X$, $\nabla_X H = -D_H X$, $\nabla_H K = 0$ for $X \in \mathfrak{g}'$, H, $K \in \mathfrak{a}$.

The following lemma is very useful.

LEMMA 3.1. *If $\mathfrak{g}'$ is abelian and $\mathrm{ad}_H|_{\mathfrak{g}'}$ is symmetric with respect to $\langle\ ,\ \rangle$ for all $H \in \mathfrak{a}$, then $\mathfrak{g}$ has sectional curvature $K \leqslant 0$ if and only if $K(X, Y) \leqslant 0$ for all X, Y in $\mathfrak{g}'$.*

PROOF. We observe that $\nabla_H = 0$, $\nabla_X H = -[H, X] \in \mathfrak{g}'$, $\nabla_X Y \in \mathfrak{a}$ for all X, $Y \in \mathfrak{g}'$, $H \in \mathfrak{a}$. Then, one computes for $X, Y \in \mathfrak{g}'$, $H, J \in \mathfrak{a}$

$$\begin{aligned} R(X+H, Y+J) &= [\nabla_{X+H}, \nabla_{Y+J}] - \nabla_{[X+H, Y+J]} \\ &= [\nabla_X, \nabla_Y] - \nabla_{[X,J]} - \nabla_{[H,Y]}, \end{aligned}$$

$$\begin{aligned} \langle R(X+H, Y+J)(Y+J), X+H\rangle &= \langle [\nabla_X, \nabla_Y]Y, X\rangle + \langle [\nabla_X, \nabla_Y]J, H\rangle \\ &\quad - \langle \nabla_{[X,J]}Y, H\rangle - \langle \nabla_{[X,J]}J, X\rangle - \langle \nabla_{[H,Y]}Y, H\rangle - \langle \nabla_{[H,Y]}J, X\rangle. \end{aligned}$$

But

$$\begin{aligned} \langle [\nabla_X, \nabla_Y]Y, X\rangle &= \langle R(X,Y)Y, X\rangle, \\ \langle [\nabla_X, \nabla_Y]J, H\rangle &= -\langle \nabla_Y J, \nabla_X H\rangle + \langle \nabla_X J, \nabla_Y H\rangle \\ &= -\langle [J, Y], [H, X]\rangle + \langle [J, X], [H, Y]\rangle \\ &= \langle [\mathrm{ad}_J, \mathrm{ad}_H]Y, X\rangle = 0 \end{aligned}$$

since $\mathfrak{a}$ is abelian, and

$$\begin{aligned} -\langle \nabla_{[X,J]}Y, H\rangle &- \langle \nabla_{[X,J]}J, X\rangle - \langle \nabla_{[H,Y]}Y, H\rangle - \langle \nabla_{[H,Y]}J, X\rangle \\ &= -\langle Y, [H, [X, J]]\rangle + \langle [J, [X, J]], X\rangle \\ &\quad - \langle Y, [H, [H, Y]]\rangle + \langle [J, [H, Y]], X\rangle \\ &= 2\langle [H, Y], [J, X]\rangle - \langle [J, X], [J, X]\rangle - \langle [H, Y], [H, Y]\rangle \\ &= -|[H, Y] - [J, X]|^2. \end{aligned}$$

Then

$$\langle R(X+H, Y+J)(Y+J), X+H\rangle = \langle R(X,Y)Y, X\rangle - |[H,Y] - [J,X]|^2$$

and consequently the lemma follows.

REMARK 3.2. If $\{\mathrm{ad}_H|_{\mathfrak{g}'}\colon H \in \mathfrak{a}\}$ are symmetric and there exists an orthonormal basis $\{H_1, \ldots, H_k\}$ of $\mathfrak{a}$ such that ad_{H_i} is semidefinite, then Lemma 3.1 provides examples of Lie algebras $\mathfrak{g}$ of sectional curvature $K \leqslant 0$. In fact, let D_i denote D_{H_i}, $i = 1, \ldots, k$. Then, since for X, Y in $\mathfrak{g}'$

$$U(X,Y) = \sum_{i=1}^{k} \langle U(X,Y), H_i\rangle H_i = -2\sum_{i=1}^{k} \langle D_i X, Y\rangle H_i,$$

it follows from the curvature formula that for X, Y orthonormal in $\mathfrak{g}'$

$$K(X,Y) = \frac{1}{4}|U(X,Y)|^2 - \frac{1}{4}\langle U(X,X), U(Y,Y)\rangle$$
$$= \sum_{i=1}^{k}\left(\langle D_i X, Y\rangle^2 - \langle D_i X, X\rangle\langle D_i Y, Y\rangle\right) \leqslant 0.$$

EXAMPLE 3.3. Let $\mathfrak{g}$ be the Lie algebra generated by $\{e_1,\dots,e_n,e_{n+1},\dots,e_{n+k}\}$, $n \geqslant k \geqslant 2$, and Lie bracket

$$[e_i,e_j] = 0, \qquad i,j = 1,\dots,n,$$
$$[e_{n+i},e_{n+j}] = 0, \qquad i,j = 1,\dots,k,$$
$$[e_{n+i},e_j] = a_i^j e_j, \qquad i = 1,\dots,k,\ j = 1,\dots,n,$$

where $a_i^j \geqslant 0$ are chosen such that $\det(a_j^i) \neq 0$, i, $j = 1,\dots,k$, and for each $j = k+1,\dots,n$, the vector $(a_1^j,\dots,a_k^j)$ is nonzero; hence $\mathfrak{g}'$ is spanned by $\{e_1,\dots,e_n\}$. Let $\langle\ ,\ \rangle$ be the inner product on $\mathfrak{g}$ such that $\langle e_i,e_j\rangle = \delta_{ij}$, $i,j = 1,\dots,n+k$. If $H \in \mathfrak{a}$, $H = \sum_{j=1}^{k}\langle H, e_{n+j}\rangle$; one computes

$$\operatorname{ad}_H(e_i) = \left(\sum_{j=1}^{k}\langle H, e_{n+j}\rangle a_j^i\right)e_i = \left\langle H, \sum_{j=1}^{k} a_j^i e_{n+j}\right\rangle e_i.$$

Then $\operatorname{ad}_H(e_i) = \lambda_i(H)e_i$ with $\lambda_i(H) = \langle H, H_i\rangle$ and $H_i = \sum_{j=1}^{k} a_j^i e_{n+j}$, $i = 1,\dots,k$. Consequently $\{\operatorname{ad}_H|_{\mathfrak{g}'}\colon H \in \mathfrak{a}\}$ are symmetric with respect to the metric $\langle\ ,\ \rangle$, $\{\operatorname{ad}_{e_{n+j}}|_{\mathfrak{g}'}\colon j = 1,\dots,k\}$ are positive semidefinite and from Remark 3.2 $\mathfrak{g}$ has sectional curvature $K \leqslant 0$.

We observe that since $\{H_i\}_{i=1}^{k}$ are independent, $\lambda_i(H) = 0$ for all $i = 1,\dots,k$ implies $H = 0$. Hence if $X = \sum_{i=1}^{n}\alpha_i e_i$ is a unit vector in $\mathfrak{g}'$ with $\alpha_i \neq 0$ for all $i = 1,\dots,n$, then from the curvature formulas given in Lemma 3.1 and Remark 3.2,

$$|H|^2 K(X,H) = -|[H,X]|^2 = -\sum_{i=1}^{n}\alpha_i^2\lambda_i(H)^2 < 0 \quad \text{for all } H \in \mathfrak{a},$$

$K(X,Y) < 0$ for all $Y \in \mathfrak{g}'$ independent of X, and up to a positive constant,

$$K(X,Y+H) = |X \wedge Y|^2 K(X,Y) + |H|^2 K(X,Y) < 0 \quad \text{for all } H \in \mathfrak{a}.$$

Therefore, the only parallel Jacobi field on the geodesic γ in G with $\gamma(0) = e$, $\gamma'(0) = X$ is γ', and consequently $\operatorname{rank}(G) = 1$, where G is the simply connected Lie group associated to $(\mathfrak{g},\langle\ ,\ \rangle)$. Moreover, since $\dim\mathfrak{a} \geqslant 2$, G does not satisfy visibility, it is irreducible and it is not symmetric since there are planes with zero curvature.

EXAMPLE 3.4. Let $\mathfrak{g}$ be a nonabelian Lie algebra of dimension n such that $\mathfrak{g}'$ is one dimensional and let $\langle\ ,\ \rangle$ be an inner product in $\mathfrak{g}$. If $\mathfrak{a}$, the orthogonal complement of $\mathfrak{g}'$ is abelian then the associated Lie group G to $(\mathfrak{g},\langle\ ,\ \rangle)$ has sectional curvature $K \leqslant 0$ and $\operatorname{rank}(G) = n-1$.

Set $\mathfrak{g}' = \mathbf{R}e_1$ with e_1 a unit vector and let $\{e_2,\dots,e_n\}$ be an orthonormal basis for $\mathfrak{a}$. Then the Lie bracket in $\mathfrak{g}$ is given by

$$[e_i,e_j] = 0, \qquad i,j = 2,\dots,n,$$
$$[e_i,e_1] = a_i e_1, \qquad i = 2,\dots,n.$$

Therefore, if $H = \sum_{i=2}^{n}\langle H, e_i\rangle e_i$ then $\operatorname{ad}_H(e_1) = \lambda(H)e_1$ where $\lambda(H) = \langle H, H_0\rangle$ and $H_0 = \sum_{i=2}^{n} a_i e_i$ is a nonzero vector in $\mathfrak{a}$. From the curvature formula given in Lemma 3.1, up to a positive constant, $K(\alpha e_1 + H, \beta e_1 + J) = -|\beta\lambda(H) - \alpha\lambda(J)|^2 \leqslant 0$ for all $H, J \in \mathfrak{a}$, $\alpha, \beta \in \mathbf{R}$.

In order to compute rank(G), we will prove that G is isometric to $\mathbf{R}^{n-2} \times H^2$. We compute $\nabla_{H_0}e_1 = 0$, $\nabla_{e_1}H_0 = -\lambda(H_0)e_1$, $\nabla_{e_1}e_1 = H_0$, $\nabla_{H_0}H_0 = 0$; consequently $\mathfrak{h}$, the Lie algebra generated by $\{e_1, H_0\}$, and $\mathfrak{h}^{\perp}$ are totally geodesic ideals of $\mathfrak{g}$ such that $[\mathfrak{h}, \mathfrak{h}^{\perp}] = 0$. Then, since $\mathfrak{h}^{\perp}$ is abelian, applying Lemma 4.1 (see §4) G is isometric to the riemannian product of $\mathbf{R}^{n-2}$ and H^2, where H^2 with the induced metric has sectional curvature $K = K(e_1, H_0) = -|H_0|^2$. Hence rank($G$) $= n - 1$.

4. Homogeneous spaces of $K \leqslant 0$ and dimension $\leqslant 4$.

LEMMA 4.1. *Let G be a simply connected Lie group with a left invariant metric and Lie algebra $\mathfrak{g} = \mathfrak{t} + \mathfrak{k}$, the orthogonal direct sum of an ideal $\mathfrak{t}$ and a subalgebra $\mathfrak{k}$ of $\mathfrak{g}$ such that $\operatorname{ad}_X|_{\mathfrak{t}}$ is skew-symmetric for all $X \in \mathfrak{k}$. If T and K denote the connected Lie subgroups of G with Lie algebras $\mathfrak{t}$ and $\mathfrak{k}$, respectively, and left invariant metrics induced by the metric of G, then G is isometric (isomorphic as Lie group) to the Riemannian product $T \times_\sigma K$, the semidirect product of T and K (the action of K on T is given by $\sigma_k(t) = ktk^{-1}$).*

Observe that in the case $\mathfrak{g}$ is the direct sum of ideas $\mathfrak{t}$ and $\mathfrak{k}$, G is the direct product TK.

PROOF. Since $\operatorname{ad}_X\colon \mathfrak{t} \to \mathfrak{t}$ is skew-symmetric for any $X \in \mathfrak{k}$, $\operatorname{Ad}(k) = (d\sigma_k)_e$ are isometries and K acts on T by isometries. By [**3**, Proposition 1] the product metric on $T \times_\sigma K$ is left invariant. Let $f\colon T \times_\sigma K \to G$ defined by $(t, k) \to t \cdot k$. It is known (see [**9**, Chapter 3]) that f is an isomorphism of Lie groups. It is also an isometry since $df_{(e,e)}(T \times_\sigma K) = \mathfrak{t} + \mathfrak{k} \to \mathfrak{g}$ preserves the inner product and both of the metrics in $T \times_\sigma K$ and G are left invariant.

Observe that if $[\mathfrak{t}, \mathfrak{k}] = 0$ then $tk = kt$ for all $t \in T$, $k \in K$ and the last assertion follows.

THEOREM 4.2. *Let G be a solvable and simply connected Lie group with a nonflat left invariant metric of $K \leqslant 0$ and* $\dim G \leqslant 4$. *Then, either* rank(G) $= 1$ *or G is one of the following riemannian products*: $G = \mathbf{R} \times T$ *where T is a two- or three-dimensional Lie group satisfying visibility and* rank(G) $= 2$,

$G = \mathbf{R}^2 \times T^2$, rank($G$) $= 3$,

$G = H^2 \times T^2$, rank(G) $= 2$,

where H^2, T^2 are two-dimensional spaces of constant curvature $K < 0$. Moreover, when rank(G) *is one, we have that if* $\dim G = 3$ *rank one and visibility are equivalent and if* $\dim G = 4$, *G satisfies visibility or not depending on whether* $\dim \mathfrak{g}'^{\perp} = 1$ *or* 2.

PROOF. Let $\mathfrak{g} = \mathfrak{g}' + \mathfrak{a}$, $\mathfrak{a}$ the orthogonal complement of $\mathfrak{g}'$ with respect to the metric $\langle\ ,\ \rangle$. We consider the following cases:

(1) $\dim \mathfrak{a} = 1$. Either G satisfies visibility and hence rank(G) $= 1$ (see Theorem 2.6) or visibility fails in G and by [**4**, Theorem 3.1] G is isometric to $\mathbf{R} \times T$, $\mathbf{R}^2 \times T$ where T is a Lie group with a left invariant metric satisfying visibility (hence

rank $T = 1$). In case $\dim T = 2$, T is a two-dimensional space of constant curvature $K < 0$.

(2) $\dim \mathfrak{a} = 2$ or 3. If $\dim \mathfrak{g}' = 1$, this situation was studied in Example 3.4 of §3. Then $G = \mathbf{R} \times H^2$ or $\mathbf{R}^2 \times H^2$, H^2 is a space of constant negative curvature, and $\operatorname{rank}(G) = 2$ or 3, respectively.

Next we consider the case $\dim \mathfrak{g}' = 2$ ($\dim \mathfrak{a} = 2$).

The complexification of $\mathfrak{g}'$, $\mathfrak{g}'^c$ decomposes $\mathfrak{g}'^c = \mathfrak{g}_0'^c + \Sigma_{\lambda \neq 0} \mathfrak{g}_\lambda'^c$ where $\mathfrak{g}_\lambda'^c = \{X \in \mathfrak{g}'^c\colon (\operatorname{ad}_H - \lambda(H)1)^k \cdot X = 0$ for some $k \geqslant 1$ and for all $H \in \mathfrak{a}\}$ is the associated root space for the root $\lambda \in \mathfrak{a}^{c*} - \{0\}$ and $\mathfrak{g}' = \Sigma_{\alpha,\beta} \mathfrak{g}'_{\alpha\beta}$ where $\mathfrak{g}'_{\alpha\beta} = \mathfrak{g}' \cap (\mathfrak{g}_\lambda'^c \oplus \mathfrak{g}_{\bar\lambda}'^c)$ if $\lambda = \alpha + i\beta$.

We observe that $\mathfrak{g}_0'^c = 0$; otherwise since 0 is a root of $\mathfrak{a}^c$, there exist $X \neq 0$ in $\mathfrak{g}'$ such that $[H, X] = 0$ for all H in $\mathfrak{a}$ implying that $X \in z(\mathfrak{g})$, the center of $\mathfrak{g}$ ($\mathfrak{g}'$ is abelian). This contradicts the fact that $z(\mathfrak{g})$ is orthogonal to $\mathfrak{g}'$ when the metric has $K \leqslant 0$.

(i) *Assume there is a complex root* $\lambda = \alpha + i\beta$. We remark first that if $\alpha(H) = 0$ for some $H \in \mathfrak{a}$, then $\operatorname{ad}_H|_{\mathfrak{g}'}$ is skew-symmetric. In fact, since $\mathfrak{g}_\lambda'^c$ and $\mathfrak{g}_{\bar\lambda}'^c$ are one dimensional and ad_H-invariant, $\operatorname{ad}_H|_{\mathfrak{g}_\lambda'^c}$, $\operatorname{ad}_H|_{\mathfrak{g}_{\bar\lambda}'^c}$ are skew-symmetric and it follows from [**1**, Lemma 4.3 and Theorem 5.2] that $\mathfrak{g}_\lambda'^c$ and $\mathfrak{g}_{\bar\lambda}'^c$ are orthogonal with respect to the complex inner product induced by $\langle\ ,\ \rangle$; therefore $\operatorname{ad}_H|_{\mathfrak{g}'}$ is skew-symmetric ($\mathfrak{g}' = \mathfrak{g}' \cap (\mathfrak{g}_\lambda'^c \oplus \mathfrak{g}_{\bar\lambda}'^c)$).

We may thus assume $\alpha \neq 0$; otherwise from the remark above $\operatorname{ad}_H|_{\mathfrak{g}'}$ is skew-symmetric for all $H \in \mathfrak{a}$ and consequently $K \equiv 0$ (see [**8**, Theorem 1.5]). Let $\alpha(H) = \langle H, H_0\rangle$ with $H_0 \neq 0$ in $\mathfrak{a}$ and let H_1 be orthogonal to H_0 in $\mathfrak{a}$; then $\alpha(H_1) = 0$ and $\operatorname{ad}_H|_{\mathfrak{g}'}$ is skew-symmetric. Moreover, since $\det(\operatorname{ad}_{H_0}) = \alpha(H_0)^2 + \beta(H_0)^2 \neq 0$ then $[H_0, \mathfrak{g}'] = \mathfrak{g}'$. Hence, setting $\mathfrak{t} = \mathfrak{g}' + \mathbf{R}H_0$, one has that $\mathfrak{t}$ is a totally geodesic ideal of $\mathfrak{g}$ such that $\operatorname{ad}_{H_1}\colon \mathfrak{t} \to \mathfrak{t}$ is skew-symmetric and $\mathfrak{t}' = [\mathfrak{t}, \mathfrak{t}] = \mathfrak{g}'$. It then follows from Lemma 4.1 that G is isometric to the riemannian product $T \times \mathbf{R}$ where T is the connected Lie subgroup of G with Lie algebra $\mathfrak{t}$. Furthermore T, with the induced metric satisfies visibility since $\operatorname{ad}_{H_0}|_{\mathfrak{t}'} = \operatorname{ad}_{H_0}|_{\mathfrak{g}'}$ has all eigenvalues with real part $\alpha(H_0) \neq 0$ (see [**4**, Theorem 2.3]).

(ii) *Assume there is a unique real root* $\lambda \in \mathfrak{a}^* - \{0\}$. In this case $\operatorname{ad}_H - \lambda(H)1$ is nilpotent for all $H \in \mathfrak{a}$, and since $\dim \mathfrak{g}' = 2$ it follows that for any $X \in \mathfrak{g}'$ $(\operatorname{ad}_H - \lambda(H)1)^2 X = 0$ for all $H \in \mathfrak{a}$.

Let $\lambda(H) = \langle H, H_0\rangle$ with $H_0 \neq 0$ in $\mathfrak{a}$ and let H_1 be orthogonal to H_0 in $\mathfrak{a}$; we will see that $\operatorname{ad}_{H_1}|_{\mathfrak{g}'} = 0$. In fact, if $Y = \operatorname{ad}_{H_1} X \neq 0$ for some X in $\mathfrak{g}'$ then $\operatorname{ad}_{H_1} Y = \operatorname{ad}^2_{H_1} X = 0$. (by the choice of H_1). Since $S_{H_1} Y = 0$ ($S_{H_1} Y = \nabla_{H_1} Y = 0$, see Remark 1.2) the orthogonal complement of $\mathbf{R}Y$ in $\mathfrak{g}'$ is ad_{H_1}-invariant, and hence

$$0 = \left\langle \left[H_1, X - \langle X, Y\rangle \frac{Y}{|Y|^2} \right], Y \right\rangle = \langle [H_1, X], Y\rangle = |Y|^2$$

which is a contradiction.

Now, since $\mathfrak{g}' = \{X \in \mathfrak{g}'\colon (\operatorname{ad}_{H_0} - \lambda(H_0)1)^2 \cdot X = 0\}$, $\operatorname{ad}_{H_0}|_{\mathfrak{g}'}$ has $\lambda(H_0) > 0$ as unique eigenvalue; hence $\det(\operatorname{ad}_{H_0}|_{\mathfrak{g}'}) = \lambda(H_0)^2 > 0$ and $[H_0, \mathfrak{g}'] = \mathfrak{g}'$. Setting $\mathfrak{t} = \mathfrak{g}' + \mathbf{R}H_0$, $\mathfrak{t}$ is a totally geodesic ideal of $\mathfrak{g}$ since $\operatorname{ad}_{H_1}|_{\mathfrak{t}} = 0$, and $\operatorname{ad}_{H_0}|_{\mathfrak{t}'}$ has all eigenvalues $\lambda(H_0) > 0$.

Then, from Lemma 4.1 G is isometric to the riemannian product $T \times \mathbf{R}$ where T, the connected Lie subgroup of G with Lie algebra $\mathfrak{t}$ and induced metric satisfies visibility.

(iii) *Assume there are two distinct real roots* λ_1, λ_2 *in* $\mathfrak{a}^* - \{0\}$. Then $\mathfrak{g}'$ is the direct sum of $\mathfrak{g}'_{\lambda_1}$ and $\mathfrak{g}'_{\lambda_2}$ where $\mathfrak{g}'_{\lambda_1} = \mathbf{R}e_1$, $\mathfrak{g}'_{\lambda_2} = \mathbf{R}e_2$ with e_1, e_2 unit vectors in $\mathfrak{g}'$ such that

$$[H, e_1] = \lambda_1(H)e_1, \quad [H, e_2] = \lambda_2(H)e_2 \quad \text{for all } H \in \mathfrak{a}.$$

Suppose $\lambda_2 = c\lambda_1$, $c \in \mathbf{R}$ and $\lambda_1(H) = \langle H, H_0 \rangle$ with $H_0 \neq 0$ in $\mathfrak{a}$. If H_1 is orthogonal to H_0 in $\mathfrak{a}$ then $\operatorname{ad}_{H_1}|_{\mathfrak{g}'} = 0$ and in the same way as in case (ii) G is isometric to $\mathbf{R} \times T$ where T is a Lie subgroup of G satisfying visibility.

Now we suppose λ_1 and λ_2 are independent in $\mathfrak{a}^*$. If H_i, $i = 1, 2$, are chosen in $\mathfrak{a}$ such that $\lambda_i(H) = \langle H, H_i \rangle$, $i = 1, 2$, then $\{H_1, H_2\}$ are independent. Take H in $\mathfrak{a}$ such that $\langle H, H_1 \rangle = 0$. Since $S_H e_1 = 0$, the one-dimensional subspace of $\mathfrak{g}'$ orthogonal to e_1 is an eigenspace for ad_H and hence it coincides with $\mathbf{R}e_2$, implying $\langle e_1, e_2 \rangle = 0$. Consequently $\{\operatorname{ad}_H|_{\mathfrak{g}'}\colon H \in \mathfrak{a}\}$ are symmetric. One computes (see Remark 3.2) $K(e_1, e_2) = -\langle H_1, H_2 \rangle$ and hence $K(X, Y) = -\langle H_1, H_2 \rangle$ for any orthonormal vectors X, Y in $\mathfrak{g}'$.

Assume $\langle H_1, H_2 \rangle \neq 0$. Since for any unit vectors X in $\mathfrak{g}'$ and $H \in \mathfrak{a}$

$$K(X, H) = -|[H, X]|^2 = -\left(\lambda_1(H)^2 \langle X, e_1 \rangle^2 + \lambda_2(H)^2 \langle X, e_2 \rangle^2\right),$$

by choosing X in $\mathfrak{g}'$ such that $\langle X, e_i \rangle \neq 0$, $i = 1, 2$, one has $K(X, H) < 0$ for all $H \in \mathfrak{a}$ and $K(X, Y) < 0$ for all Y orthogonal to X in $\mathfrak{g}'$. Therefore if γ is the geodesic in G with $\gamma(0) = e$, $\gamma'(0) = X$ then $\operatorname{rank}(\gamma) = 1$ and G has rank one. Since $\dim \mathfrak{a} = 2$, G does not satisfy visibility (see [**4**, Theorem 2.4]).

Now consider the case $\langle H_1, H_2 \rangle = 0$. Then $\{e_1, e_2, H_1, H_2\}$ is an orthogonal basis for $\mathfrak{g}$ such that $[H_1, e_2] = 0 = [H_2, e_1]$. If $\mathfrak{t}_i$ denotes the Lie subalgebra of $\mathfrak{g}$ generated by $\{e_i, H_i\}$, $i = 1, 2$, $\mathfrak{t}_1$ and $\mathfrak{t}_2$ are ideals totally geodesic in $\mathfrak{g}([\mathfrak{t}_1, \mathfrak{t}_2] = 0)$, such that $\mathfrak{g} = \mathfrak{t}_1 + \mathfrak{t}_2$ is an orthogonal direct sum. Hence Lemma 4.1 implies that G is isometric to the riemannian product $T_1 \times T_2$ where T_i is the connected Lie subgroup of G with Lie algebra $\mathfrak{t}_i$ and the metric in T_i is the one induced by G. Since $K(e_i, H_i) = -|H_i|^2$ it follows that T_i is a two-dimensional space of constant curvature $K < 0$.

By examining all the cases and from the fact that in dimension $\leqslant 3$ visibility and rank one are equivalent (Corollary 2.5) Theorem 4.2 follows.

REMARK 4.3. The three-dimensional Lie groups G with left invariant metrics of $K \leqslant 0$ satisfying visibility are determined by the nonunimodular Lie algebras $\mathfrak{g}$ of dimension three (see [**8**, §6]) given by an orthonormal basis $\{e_1, e_2, e_3\}$ such that

$$[e_2, e_3] = 0, \quad [e_1, e_2] = \alpha e_2 + \beta e_3, \quad [e_1, e_3] = \gamma e_2 + \delta e_3,$$

with α, β, γ, δ satisfying $\alpha \geqslant \delta > 0$, $\alpha\gamma + \beta\delta = 0$, $4\alpha\delta > (\beta + \delta)^2$ and $4(\gamma^2 + \delta^2) \geqslant (\beta - \gamma)^2$.

In fact, one computes $K(e_2, e_3) = \frac{1}{4}(\beta + \gamma)^2 - \alpha\delta$, $-K(e_1, e_2) = \alpha^2 + \beta^2 - \frac{1}{4}(\beta - \gamma)^2$, $-K(e_1, e_3) = \gamma^2 + \delta^2 - \frac{1}{4}(\beta - \gamma)^2$. Therefore, since $-K(e_1, e_2)$, $-K(e_1, e_3)$ are the eigenvalues of $S^2 + BA$ and $K(\mathfrak{g}') = K(e_2, e_3) < 0$, $\mathfrak{g}$ has $K \leqslant 0$; clearly $\mathfrak{g}$ satisfies visibility by observing that D is positive definite.

COROLLARY 4.4. *Let H be a simply connected homogeneous space of nonpositive curvature and* $\dim H \leqslant 4$. *Then either* $\operatorname{rank}(H) = 1$ *or H is one of the following riemannian products*:

$H = \mathbf{R} \times T$, *T a rank one homogeneous space satisfying visibility*; $\operatorname{rank}(H) = 2$,

$H = \mathbf{R}^2 \times T^2$, $\operatorname{rank}(H) = 3$,

$H = H^2 \times T^2$, $\operatorname{rank}(H) = 2$, *where T^2, H^2 are two-dimensional spaces of constant negative curvature*.

Moreover, if $\dim H = 3$ *then rank one, visibility and without de Rham flat factor are equivalent. If* $\dim H = 4$ *and H has no de Rham flat factor then either* $\operatorname{rank}(H) = 1$ *or H is the riemannian product of two 2-dimensional spaces of negative curvature (in whose case H is a symmetric space).*

COROLLARY 4.5. *Let H be a simply connected homogeneous space of nonpositive curvature. If H is irreducible then either H has rank one or* $\dim H \geqslant 5$.

Note that this bound for $\dim H$ is the best possible since $SL(3, \mathbf{R})/SO(3)$ is an irreducible symmetric space of rank two and dimension five.

REFERENCES

1. R. Azencott and E. Wilson, *Homogeneous manifolds with negative curvature*. I, Trans. Amer. Math. Soc. **215** (1976), 323-362.

2. W. Ballmann, M. Brin and P. Eberlein, *Structure of manifolds of nonpositive curvature*. I, Ann. of Math. **122** (1985).

3. L. Bérard-Bergery, *Sur la courbure des métriques riemanniennes invariantes des groupes de Lie et des espaces homogènes*, Ann. Sci. Ecole Norm. Sup. (4) **11** (1978), 543–576.

4. M. J. Druetta, *Homogeneous Riemannian manifolds and the visibility axiom*, Geom. Dedicata **17** (1985), 239–251.

5. P. Eberlein and B. O'Neill, *Visibility manifolds*, Pacific J. Math. **46** (1973), 45–109.

6. E. Heintze, *On homogeneous manifolds of negative curvature*, Math. Ann. **211** (1974), 23–34.

7. S. Helgason, *Differential geometry, Lie groups, and symmetric spaces*, Academic Press, New York, 1978.

8. J. Milnor, *Curvatures of left invariant metrics on Lie groups*, Adv. in Math. **21** (1976), 293–329.

9. V. S. Varadarajan, *Lie groups, Lie algebras and their representations*, Prentice-Hall, Englewood Cliffs, N.J., 1974.

FACULTAD DE MATEMÁTICA, ASTRONOMÍA Y FÍSICA, UNIVERSIDAD NACIONAL DE CÓRDOBA, VALPARAÍSO Y ROGELIO MARTÍNEZ, CIUDAD UNIVERSITARIA, 5.000 CÓRDOBA, ARGENTINA